MATH 2060 Mathematical Analysis II
midterm suggested solution
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Q1 Let f be a Riemann integrable function defined on $[a,b]$.

(a) Show that the square function f^2 is also integrable function on $[a,b]$.

Proof. Since $f \in R[a,b]$, there exists $M > 0$ such that

$$|f(x)| \le M \quad , \forall x \in [a,b].$$

Let $\epsilon > 0$, there exists $\delta > 0$ such that for all partition $P : a = x_1 < x_2 < ... < x_{n+1} = b$ at which $||P|| < \delta$, we have

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} w_i(f)\Delta x_i < \frac{\epsilon}{2M}$$

where $w_i(f) = \sup\{|f(x) - f(y)| : x, y \in [x_i, x_{i+1}]\}$.

Noted that $w_i(f) = \sup\{|f(x) - f(y)| : x, y \in [x_i, x_{i+1}]\}$=$\sup\{f(x) - f(y) : x, y \in [x_i, x_{i+1}]\}$. For all $x, y \in [x_i, x_{i+1}]$,

$$f(x) - f(y) \le \sup\{f(a) - f(b) : a, b \in [x_i, x_{i+1}]\}$$
$$f(y) - f(x) \le \sup\{f(a) - f(b) : a, b \in [x_i, x_{i+1}]\}$$

Thus,

$$\sup\{|f(x) - f(y)| : x, y \in [x_i, x_{i+1}]\} \le \sup\{f(x) - f(y) : x, y \in [x_i, x_{i+1}]\}.$$

On the other hand,

$$f(x) - f(y) \le \sup\{|f(a) - f(b)| : a, b \in [x_i, x_{i+1}]\} \quad \forall x, y \in [x_i, x_{i+1}].$$

So, $w_i(f) = \sup\{|f(x) - f(y)| : x, y \in [x_i, x_{i+1}]\}$=$\sup\{f(x) - f(y) : x, y \in [x_i, x_{i+1}]\}$.

On each $[x_i, x_{i+1}]$, for all $x, y \in [x_i, x_{i+1}]$,

$$|f^2(x) - f^2(y)| \le |f(x) + f(y)||f(x) - f(y)| \le 2Mw_i(f).$$

Thus, $w_i(f^2) \le 2Mw_i(f)$ for all $i = 1, 2, ...n$. So there exists $\delta > 0$ such that for all partition $P : a = x_1 < x_2 < ... < x_{n+1} = b$ at which $||P|| < \delta$, we have

$$U(f^2,P) - L(f^2,P) = \sum_{i=1}^{n} w_i(f^2)\Delta x_i < 2M \sum_{i=1}^{n} w_i(f)\Delta x_i < \epsilon.$$

□

(b) Show that if there exists $\delta > 0$ such that $|f| \ge \delta$, then $\sqrt{|f|}$ is integrable.

Proof. Let $\epsilon > 0$. Since $f \in R[a,b]$, there exists partition P such that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} w_i(f)\Delta x_i < 2\sqrt{\delta}\epsilon.$$

On each $[x_i, x_{i+1}]$, for any $x, y \in [x_i, x_{i+1}]$,

$$|\sqrt{|f(x)|} - \sqrt{|f(y)|}| \leq \frac{|f(x) - f(y)|}{\sqrt{|f(y)|} + \sqrt{|f(x)|}} \leq \frac{1}{2\sqrt{\delta}} w_i(f).$$

Thus,

$$\begin{aligned} U(\sqrt{|f|}, P) - L(\sqrt{|f|}, P) &= \sum_{i=1}^{n} w_i(\sqrt{|f|})\Delta x_i \\ &< \frac{1}{2\sqrt{\delta}} \sum_{i=1}^{n} w_i(f)\Delta x_i < \epsilon. \end{aligned}$$

□

Q2 Determine whether the following improper integrals exist:

(a) $\int_0^1 \sin x/\sqrt{x^3}\, dx$

Proof. The integral exists. We prove our claim using cauchy criterion. Let $\epsilon > 0$, there exists $\delta = \dfrac{\epsilon^2}{4} > 0$ such that for all $b > a$ and $a, b \in (0, \delta) \cap [0, 1]$,

$$\begin{aligned} \left| \int_a^b \frac{\sin x}{\sqrt{x^3}}\, dx \right| &\leq \int_a^b \frac{1}{\sqrt{x}} dx \\ &= 2\sqrt{b} - 2\sqrt{a} < 2\sqrt{\delta} = \epsilon. \end{aligned}$$

The first inequality follows from the fact that $\sin x \leq x$ for all $x \geq 0$. □

(b) $\int_1^\infty \log x/\sqrt{x^5}\, dx$

Proof. The integral exists. We prove our claim using comparsion test. Since $\log x \leq x$ for all $x \geq 1$, we have

$$0 \leq \frac{\log x}{\sqrt{x^5}} \leq \frac{1}{\sqrt{x^3}} \quad , \quad \forall x \geq 1.$$

It remains to check that $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^3}}$ exists. For all $p > 1$,

$$\int_1^p \frac{1}{\sqrt{x^3}} = 2\left(1 - \frac{1}{\sqrt{p}}\right) \to 2 \text{ as } p \to \infty.$$

□

Q3 Let f be a function defined on $(-c, c)$ for some $c > 0$.

(a) If $|f(x)| \leq |x|^\alpha$ for some $\alpha > 1$, show that $f'(0)$ exists and find it.

Proof. Since $|f(x)| \le |x|^\alpha$ for some $\alpha > 1$, we have $f(0) = 0$. Let $\epsilon > 0$, there exists $\delta = \epsilon^{\frac{1}{\alpha-1}} > 0$ such that for all $0 < |x| < \delta$,

$$\left|\frac{f(x) - f(0)}{x}\right| = |\frac{f(x)}{x}| \le |x|^{\alpha-1} < \delta^{\alpha-1} = \epsilon.$$

Thus $f'(0)$ exists and equals to 0. □

(b) Does part (a) hold when $\alpha = 1$?
Ans: No. Take $f(x) = |x|$. The assumption clearly holds. But f is not differentiable at $x = 0$.

Q4 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose that $a < b$ and $f'(a) < f'(b)$.

(a) Show that if $f'(a) < \lambda < f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = \lambda$.

Proof. Define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - \lambda x$. g is differentiable on $[a, b]$ and $g'(x) = f'(x) - \lambda$ for all $x \in [a, b]$. Since g is differentiable, so g is continuous. By Max-Min theorem, there exists $c \in [a, b]$ such that $g(c) \le g(x)$ for all $x \in [a, b]$.
Since $g'(a) < 0$, there exists $\delta > 0$ such that for all $x \in (a, a + \delta)$,

$$\frac{g(x) - g(a)}{x - a} < 0 \implies g(x) < g(a).$$

Similarly, there exists $\delta > 0$ such that for all $x \in (b - \delta, b)$,

$$\frac{g(b) - g(x)}{b - x} > 0 \implies g(x) < g(b).$$

Thus $c \in (a, b)$. Argue as above, we can deduce that $g'(c)$ can't be positive or negative. Thus, $g'(c) = 0$ which implies $f'(c) = \lambda$. □

(b) By using part (a) or otherwise, show that if $f'(x)$ is increasing on (a, b), then f' is continuous on (a, b).

Proof. Since f' is increasing, $\lim_{x \to c+} f'(x)$ and $\lim_{x \to c-} f'(x)$ exists for all $c \in (a, b)$.

One may verify this using sequential criterion. Let $\{x_n\}$ be a sequence of real numbers such that $x_n > c$ and $\lim_n x_n = c$. Since f' is increasing, $\{f'(x_n)\}$ converges by monotone convergence theorem. Thus, the limit is unique. The existence of left hand limit is proved analogously.

By the monotonic increasing property of f', we have

$$\lim_{x \to c+} f'(x) \ge f'(c) \ge \lim_{x \to c-} f'(x) \text{ for all } c \in (a, b).$$

Assume $\lim_{x \to c+} f'(x) = \alpha > f'(c)$ for some $c \in (a, b)$. By the definition of limit, there exists $\delta > 0$ such that for all $x \in (c, c + \delta)$,

$$f'(c) < \frac{f'(c) + \alpha}{2} < f'(x).$$

By result of (a), there exists $d \in (c, c + \delta)$ such that

$$f'(d) = \frac{f'(c) + \alpha}{2}.$$

Contradiction arised. So, $\lim_{x \to c+} f'(x) = f'(c) \ ,\forall c \in (a, b)$. Similarly, $\lim_{x \to c-} f'(x) = f'(c) \ ,\forall c \in (a, b)$. Hence f' is continuous on (a, b). □